جامعة بنها

كلية الحاسبات والذكاء الإصطناعي

Discrete Mathematics

Lecture 07

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Chapter 9: Relations

- Relations and Their Properties.
- Representing Relations.
- Closures of Relations.
- Equivalence Relations.
- Partial Orderings.

Introduction (1/2)

Relationships between elements of sets are represented using the structure called a **relation**, which is just a subset of the Cartesian product of the sets.

In mathematics, we study relationships such as those between a *positive integer and one that it divides*, *an integer and one that it is congruent to modulo 5*, *a real number and one that is larger than it*, *a real number* x *and the value* $f(x)$ *where* f *is a function*, and so on.

Introduction (2/2)

The most direct way to express a relationship between elements of two sets is to use ordered pairs made up of two related elements. For this reason, sets of ordered pairs are called binary relations.

Relations and Their Properties (2/30)

Definition 1:

Let A and B be sets. A *binary relation* from A to B is a subset of $A \times B$.

A *binary relation* from A to B is a set R of ordered pairs where the first element of each ordered pair comes from A and the second element comes from B.

We use the notation $a\,R\,b$ to denote that $(a, b) \in R$ and $a \not R\, b$ to denote that $(a, b) \notin R$. Moreover, when (a, b) belongs to R, a is said to be related to b by R.

Relations and Their Properties (3/30)

Example 1:

Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$.

Then $\{(\mathbf{0}, \boldsymbol{a}), (\mathbf{0}, \boldsymbol{b}), (\mathbf{1}, \boldsymbol{a}), (\mathbf{2}, \boldsymbol{b})\}$ is a relation from A to B.

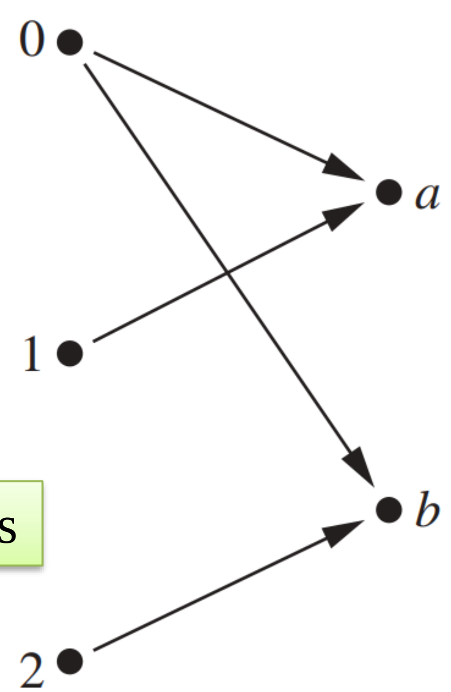
Roster notation (Roster form of set):

$$R = \{(0, a), (0, b), (1, a), (2, b)\}$$

Example 1:

Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$.

Then $\{(\mathbf{0}, \boldsymbol{a}), (\mathbf{0}, \boldsymbol{b}), (\mathbf{1}, \boldsymbol{a}), (\mathbf{2}, \boldsymbol{b})\}$ is a relation from A to B.



Using arrows

R	a	b
0	×	×
1	×	
2		×

Using table

Functions as Relations

Recall that a function f from a set A to a set B assigns exactly one element of B to each element of A. The graph of f is the set of ordered pairs (a, b) such that $b = f(a)$. Because the graph of f is a subset of $A \times B$, it is a relation from A to B.

Relations and Their Properties (5/30)

Relations on a Set

Definitions:

- A relation on the set A is a relation from A to A. In other words, a relation on a set A is a subset of $A \times A$.
- The identity relation I_A on a set A is the set $\{(a, a) | a \in A\}$
  - Ex. If $A = \{1, 2, 3\}$, then $I_A = \{(1, 1), (2, 2), (3, 3)\}$

Relations and Their Properties (6/30)

Example 2:

Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) | a \text{ divides } b\}$?

Relations and Their Properties (6/30)

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Set builder notation:

$R = \{(a, b) | a \text{ divides } b\}$

Example 2:

Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) | a \text{ divides } b\}$?

May change to be:

$a = b$
$a > b$
$a < b$
...

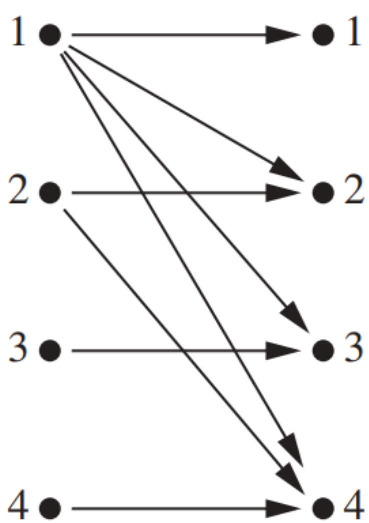
Relations and Their Properties (6/30)

Example 2:

Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) | a \text{ divides } b\}$?

Solution:

$$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$$



R	1	2	3	4
1	×	×	×	×
2		×		×
3			×	
4				×

Relations and Their Properties (7/30)

Example 3:

Let A be the set $\{-1, 0, 1, 2\}$. Which ordered pairs are in the following relations:

$R_1 = \{(a\,, b)|a < b\}$

$R_2 = \{(a\,, b)|a > b\}$

$R_3 = \{(a\,, b)|a = b\}$

$R_4 = \{(a\,, b)|a = -b\}$

$R_5 = \{(a\,, b)|a = b \text{ or } a = -b\}$

$R_6 = \{(a\,, b)|0 \leq a + b \leq 1\}$

Example 3: **Solution:**

Let A be the set $\{-1, 0, 1, 2\}$. Which ordered pairs are in the following relations:

$$R_1 = \{(a\,, b) | a < b\}$$

$$= \{(-1, 0), (-1, 1), (-1, 2), (0, 1), (0, 2), (1, 2)\}$$

Relations and Their Properties (7/30)

Example 3: **Solution:**

Let A be the set $\{-1, 0, 1, 2\}$. Which ordered pairs are in the following relations:

$$R_2 = \{(a\ , b)|a > b\}$$

$$= \{(0, -1), (1, 0), (1, -1), (2, 1), (2, 0), (2, -1)\}$$

Relations and Their Properties (7/30)

Example 3: **Solution:**

Let A be the set $\{-1, 0, 1, 2\}$. Which ordered pairs are in the following relations:

$$R_3 = \{(a\ , b) | a = b\}$$

$$= \{(-1, -1), (0, 0), (1, 1), (2, 2)\}$$

Relations and Their Properties (7/30)

Example 3: **Solution:**

Let A be the set $\{-1, 0, 1, 2\}$. Which ordered pairs are in the following relations:

$$R_4 = \{(a\,, b) | a = -b\}$$

$$= \{(-1, 1), (0, 0), (1, -1)\}$$

Relations and Their Properties (7/30)

Example 3: **Solution:**

Let A be the set $\{-1, 0, 1, 2\}$. Which ordered pairs are in the following relations:

$$R_3 = \{(a\,, b)|a = b\} = \{(-1, -1), (0, 0), (1, 1), (2, 2)\}$$

$$R_4 = \{(a\,, b)|a = -b\} = \{(-1, 1), (0, 0), (1, -1)\}$$

$$R_5 = \{(a\,, b)|a = b \text{ or } a = -b\}$$

$$= \{(-1, -1), (0, 0), (1, 1), (2, 2), (-1, 1), (1, -1)\}$$

Example 3: **Solution:**

Let A be the set $\{-1, 0, 1, 2\}$. Which ordered pairs are in the following relations:

$$R_6 = \{(a\ , b) | 0 \leq a + b \leq 1\}$$

$$= \{(-1, 1), (-1, 2), (0, 0), (0, 1), (1, -1), (1, 0), (2, -1)\}$$

Example 4:

How many relations are there on a set with n elements?

It is not hard to determine the number of relations on a finite set, because a relation on a set A is simply a **subset** of $A \times A$.

Note: $|A \times A| = |A|^2 = n^2$

Example 4:

How many relations are there on a set with n elements?

It is not hard to determine the number of relations on a finite set, because a relation on a set A is simply a **subset** of $A \times A$.

Note: $|A \times A| = |A|^2 = n^2$

Solution:

A relation on a set A is a subset of $A \times A$. Because $A \times A$ has n^2 elements when A has n elements, there are 2^{n^2} subsets of $A \times A$.

Properties of Relations

There are several properties that are used to classify relations on a set. We will introduce the most important of these relations.

- Reflexive
- Irreflexive
- Symmetric
- Antisymmetric
- Transitive

Reflexive and Irreflexive

A relation R on a set A is called ***reflexive*** if $(a, a) \in R$ for every element $a \in A$.

A relation R on a set A is called ***irreflexive*** if $(a, a) \notin R$ for every element $a \in A$.

not reflexive ≠ irreflexive

Example 1:

Consider the following relations on $\{1, 2, 3, 4\}$ are reflexive or irreflexive or not?

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$$
$$R_2 = \{(1, 1), (1, 2), (2, 1)\},$$
$$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},$$
$$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\},$$
$$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\},$$
$$R_6 = \{(3, 4)\}.$$

Example 1:

Consider the following relations on $\{1, 2, 3, 4\}$ are reflexive or irreflexive or not?

Solution: R_3 and R_5 are reflexive

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,

$R_2 = \{(1, 1), (1, 2), (2, 1)\}$,

$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,

$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$,

$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$,

$R_6 = \{(3, 4)\}$.

Example 1:

Consider the following relations on $\{1, 2, 3, 4\}$ are reflexive or irreflexive or not?

Solution: R_3 and R_5 are reflexive | R_4 and R_6 are irreflexive

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,

$R_2 = \{(1, 1), (1, 2), (2, 1)\}$,

$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,

$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$,

$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$,

$R_6 = \{(3, 4)\}$.

Example 1:

Consider the following relations on $\{1,2,3,4\}$ are reflexive or irreflexive or not?

Solution:

R_3 and R_5 are reflexive

R_4 and R_6 are irreflexive

R_1 and R_2 are
Not reflexive
Not irreflexive

$R_1 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$,

$R_2 = \{(1,1),(1,2),(2,1)\}$,

$R_3 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}$,

$R_4 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$,

$R_5 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$,

$R_6 = \{(3,4)\}$.

Example 2:

Is the "divides" relation on the set of positive integers reflexive?

Example 2:

Is the "divides" relation on the set of positive integers reflexive?

Solution:

Because $a|a$ whenever a is a positive integer, the "divides" relation is **reflexive**.

Example 3:

Is the "divides" relation on the set of integers reflexive?

Example 3:

Is the "divides" relation on the set of integers reflexive?

Solution:

The relation is **not reflexive** because 0 does not divide 0.

Example 4:

Is the following relations on the integers are reflexive or not?

$R_1 = \{(a, b) | a \leq b\}$

$R_2 = \{(a, b) | a > b\}$

$R_3 = \{(a, b) | a = b\}$

$R_4 = \{(a, b) | a = b + 1\}$

$R_5 = \{(a, b) | a = b \text{ or } a = -b\}$

$R_6 = \{(a, b) | a + b \leq 3\}$

Example 4:

Is the following relations on the integers are reflexive or not?

Solution: R_1, R_3, and R_5 are reflexive

$R_1 = \{(a, b) | a \leq b\}$

$R_2 = \{(a, b) | a > b\}$

$R_3 = \{(a, b) | a = b\}$

$R_4 = \{(a, b) | a = b + 1\}$

$R_5 = \{(a, b) | a = b \text{ or } a = -b\}$

$R_6 = \{(a, b) | a + b \leq 3\}$

Example 4:

Is the following relations on the integers are reflexive or not?

Solution: R_1, R_3, and R_5 are reflexive — R_2, R_4, and R_6 are not reflexive

$R_1 = \{(a, b) | a \leq b\}$

$R_2 = \{(a, b) | a > b\}$ (Counter example, $2 \ngtr 2$)

$R_3 = \{(a, b) | a = b\}$

$R_4 = \{(a, b) | a = b + 1\}$ (Counter example, $2 \neq 2 + 1$)

$R_5 = \{(a, b) | a = b \text{ or } a = -b\}$

$R_6 = \{(a, b) | a + b \leq 3\}$ (Counter example, $2 + 2 \nleq 3$)

Relations and Their Properties (15/30)

Symmetric and Antisymmetric

A relation R on a set A is called ***symmetric***
if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a , b \in A$.

A relation R on a set A such that for all $a , b \in A$,
if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$
is called ***antisymmetric***.

Example 4:

Which of the following relations are symmetric and which are antisymmetric?

$$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\},$$

$$R_2 = \{(1,1), (1,2), (2,1)\},$$

$$R_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\},$$

$$R_4 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\},$$

$$R_5 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\},$$

$$R_6 = \{(3,4)\}.$$

$$R_7 = \{(1,1), (2,2)\}.$$

Example 4:

Which of the following relations are symmetric and which are antisymmetric?

Solution:

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,

$R_2 = \{(1, 1), (1, 2), (2, 1)\}$, **symmetric**

$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$, **symmetric**

$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$, **antisymmetric**

$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$, **antisymmetric**

$R_6 = \{(3, 4)\}$. **antisymmetric**

$R_7 = \{(1,1), (2,2)\}$. **symmetric and antisymmetric**

Example 5:

Is the "divides" relation on the set of positive integers symmetric?

Relations and Their Properties (17/30)

Example 5:

Is the "divides" relation on the set of positive integers symmetric?

Solution:

This relation is **not symmetric** because $1 \mid 2$, $2 \nmid 1$.

Example 6:

Is the "divides" relation on the set of positive integers antisymmetric?

Example 6:

Is the "divides" relation on the set of positive integers antisymmetric?

Solution:

This relation is **antisymmetric**.

To see this, note that if a and b are positive integers with $a \mid b$ and $b \mid a$, then $a = b$.

Relations and Their Properties (19/30)

Transitive

A relation R on a set A is called ***transitive***
If whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$,
for all $a, b, c \in A$

Relations and Their Properties (20/30)

Example 1:

Which of the following relations are transitive?

$$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\},$$
$$R_2 = \{(1,1), (1,2), (2,1)\},$$
$$R_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\},$$
$$R_4 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\},$$
$$R_5 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\},$$
$$R_6 = \{(3,4)\}.$$
$$R_7 = \{(1,1), (2,2)\}.$$

Example 1:

Which of the following relations are transitive?

Solution:

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,

$R_2 = \{(1, 1), (1, 2), (2, 1)\}$,

$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,

$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$, **transitive**

$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$, **transitive**

$R_6 = \{(3, 4)\}$. **transitive**

$R_7 = \{(1,1), (2,2)\}$. **transitive**

Example 2:

Is the "divides" relation on the set of positive integers transitive?

Example 2:

Is the "divides" relation on the set of positive integers transitive?

Solution:

This relation is **transitive.**

Suppose that a divides b and b divides c. Then there are positive integers k and l such that $b = ak$ and $c = bl$.

Hence, $c = (ak)l = a(kl)$, so a divides c.

It follows that this relation is **transitive**.

Relations and Their Properties (22/30)

Notes:

If $A = \emptyset$, then the empty relation R on the set A is ***reflexive, symmetric***, and ***transitive*** **vacuously**.

For any set A, if the relation R on the set A is empty set,

i.e., $R = \emptyset$,

then it is ***irreflexive, transitive, symmetric***, and ***antisymmetric.***

For any set A, if the relation R on the set A is universal set,

i.e., $R = U = A \times A$,

then it is ***Reflexive, transitive***, and ***symmetric.***

Combining Relations

The relations

$R_1 = \{(1,1),(2,2),(3,3)\}$ and

$R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$

can be combined to obtain

$R_1 \cup R_2 =$

$R_1 \cap R_2 =$

$R_1 - R_2 =$

$R_2 - R_1 =$

$R_1 \oplus R_2 = R_1 \cup R_2 - R_1 \cap R_2 =$

Combining Relations

The relations
$R_1 = \{(1,1),(2,2),(3,3)\}$ and
$R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$
can be combined to obtain

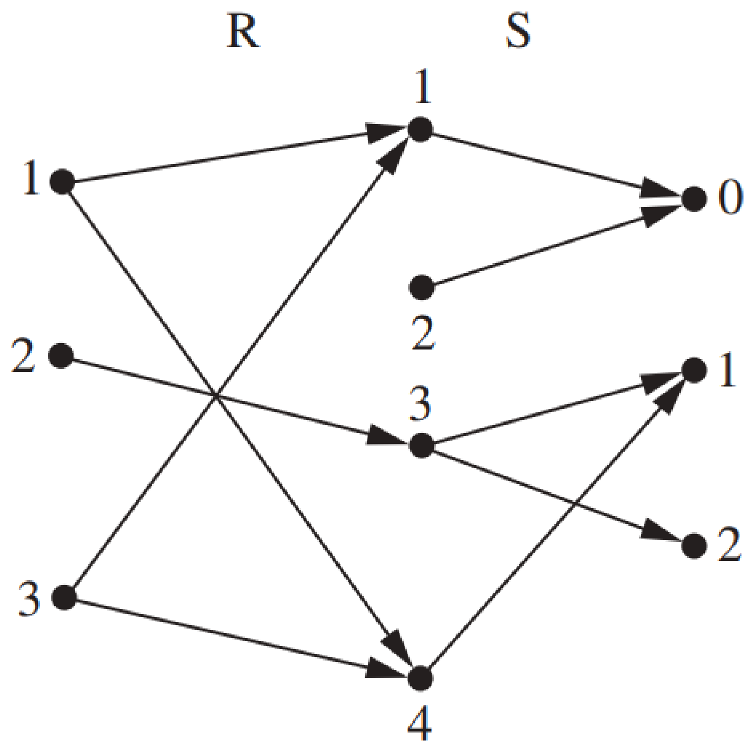
Solution:

$R_1 \cup R_2 = \{(1,1),(2,2),(3,3),(1,2),(1,3),(1,4)\}$
$R_1 \cap R_2 = \{(1,1)\}$
$R_1 - R_2 = \{(2,2),(3,3)\}$
$R_2 - R_1 = \{(1,2),(1,3),(1,4)\}$
$R_1 \oplus R_2 = R_1 \cup R_2 - R_1 \cap R_2$
$= \{(2,2),(3,3),(1,2),(1,3),(1,4)\}$

Definition – Composite (1/2)

Let R be a relation from a set A to a set B and S a relation from B to a set C. The ***composite*** of R and S is the relation consisting of ordered pairs $(a\ ,c)$, where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b\ ,c) \in S$. We denote **the composite of $\boldsymbol{R}$ and $\boldsymbol{S}$** by $\boldsymbol{S \circ R}$.

Definition – Composite (2/2)



$S \circ R$

$1 \to 1 \to 0$	$(1, 0)$
$1 \to 4 \to 1$	$(1, 1)$
$2 \to 3 \to 1$	$(2, 1)$
$2 \to 3 \to 2$	$(2, 2)$
$3 \to 1 \to 0$	$(3, 0)$
$3 \to 4 \to 1$	$(3, 1)$

Example 1:

What is the composite of the relations R and S,

where R is the relation from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$ with

$R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$ and

S is the relation from $\{1, 2, 3, 4\}$ to $\{0, 1, 2\}$ with

$S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$?

Relations and Their Properties (27/30)

Example 1:

What is the composite of the relations R and S,

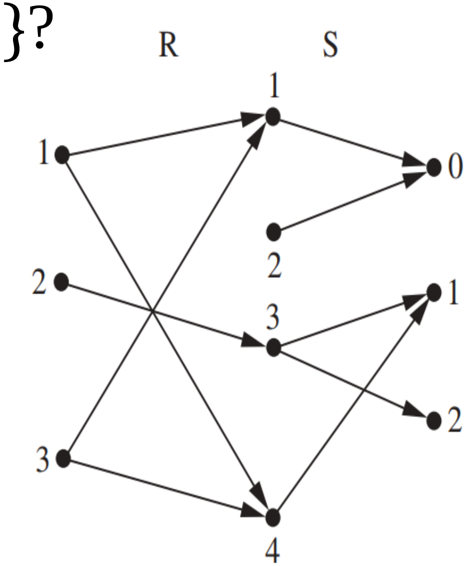
where R is the relation from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$ with

$R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$ and

S is the relation from $\{1, 2, 3, 4\}$ to $\{0, 1, 2\}$ with

$S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$?

<u>Solution:</u>



$1 \to 1 \to 0$ $(1, 0)$

$1 \to 4 \to 1$ $(1, 1)$

$2 \to 3 \to 1$ $(2, 1)$

$2 \to 3 \to 2$ $(2, 2)$

$3 \to 1 \to 0$ $(3, 0)$

$3 \to 4 \to 1$ $(3, 1)$

Example 1:

What is the composite of the relations R and S,

where R is the relation from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$ with

$R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$ and

S is the relation from $\{1, 2, 3, 4\}$ to $\{0, 1, 2\}$ with

$S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$?

Solution:

$S \circ R = \{(1, 0), (1, 1), (2, 1), (2, 2), (3, 0), (3, 1)\}$

Definition – Powers

Let R be a relation on the set A.

The powers R^n , $n = 1, 2, 3, \ldots,$ are defined recursively by

$$R^1 = R \text{ and } R^{n+1} = R^n \circ R$$

The definition shows that $R^2 = R \circ R$, $R^3 = R^2 \circ R$, and so on.

Example 2:

Let $R = \{(1,1),(2,1),(3,2),(4,3)\}$.

Find the powers R^n, $n = 2, 3, 4, \dots$

Example 2:

Let $R = \{(1,1),(2,1),(3,2),(4,3)\}$.

Find the powers R^n, $n = 2, 3, 4, \ldots$

Solution:

$$R^2 = R \circ R = \{(1,1),(2,1),(3,1),(4,2)\}$$
$$R^3 = R^2 \circ R = \{(1,1),(2,1),(3,1),(4,1)\}$$
$$R^4 = R^3 \circ R = \{(1,1),(2,1),(3,1),(4,1)\}$$

R^4 is the same R^3, it is also following that $R^n = R^3$, $n = 5, 6, 7, \ldots$

Video Lectures

All Lectures: https://www.youtube.com/playlist?list=PLxIvc-MGOs6gZlMVYOOEtUHJmfUquCjwz

Lectures #7: https://www.youtube.com/watch?v=jXHDsYVrhlY&list=PLxIvc-MGOs6gZlMVYOOEtUHJmfUquCjwz&index=48

https://www.youtube.com/watch?v=57tfpLsEDAY&list=PLxIvc-MGOs6gZlMVYOOEtUHJmfUquCjwz&index=49

Thank You

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